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AUTHOR(S):

Osaka, Hiroyuki

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Introduction to the classification of purely infinite simple C^* -algebras - Kirchberg 氏の仕事の紹介-

琉球大学・理 大坂博幸 (Hiroyuki Osaka)

1 Introduction

In this note we try to present the framework of the following classification theory of purely infinite simple C^* -algebras by Kirchberg [17].

Theorem B. If A and B are purely infinite simple, separable, unital, nuclear C^* -algebras (*pi-sun* algebras) with $A = A^{st}$ and $B = B^{st}$, then for every KK -equivalence $z \in KK(A, B)$ there exists a unital $*$ -isomorphism h from A onto B which induces the equivalent KK -element to z , where A^{st} denotes the Cuntz standard form, that is, the K_0 element $[1_A]_0$ is a zero element in $K_0(A)$.

Since any purely infinite simple C^* -algebra is stable isomorphic to some purely infinite simple unital C^* -algebra in Cuntz standard form, we get

Corollary. Let A and B be *pi-sun* algebras.

(1) A and B are KK -equivalent if and only if they are stable isomorphic.

(2) If there exists a KK -equivalence x in $KK(A, B)$ with

$$\gamma_0(x)([1_A]_0) = [1_B]_0,$$

then A and B are isomorphic, where γ_0 is a natural map from $KK(A, B)$ to $Hom(K_0(A), K_0(B))$ which is induced by Kasparov product.

Therefore,

$$(K_0(A), [1_A]_0, K_1(A))$$

is a complete system of invariant in the classification of *pi-sun* algebras satisfying the Universal Coefficient Theorem for their KK-Theory.

From this classification theorem we obtain that

- 1) For every separable simple unital nuclear C*-algebra A

$$A \otimes O_2 \cong O_2.$$

- 2) A separable simple unital nuclear C*-algebra A is purely infinite if and only if $A \cong A \otimes O_\infty$.

Now we shall look through Kirchberg's approach to Theorem B

Kirchberg proved Theorem B using his deepest result of the following characterization of exact C*-algebras [17]:

Theorem A.

- (1) A separable C*-algebra A is isomorphic to a C*-subalgebra of O_2 if and only if A is exact.

- (2) A separable unital C*-algebra A is isomorphic to the range of a conditinal expectation from O_2 onto a C*-subalgebra of O_2 if and only if A is nuclear.

Using Theorem A (1) Kirchberg defined a semigroup $EK(A, B)$ constructed by *-monomorphisms from A into $M(C_0(\mathbf{R}_+) \otimes B)/C_0(\mathbf{R}_+) \otimes B$, and show that the Grohtendieck group of $EK(A, B)$ is isomorphic to $KK(A, B)$. Next, he defines a natural semigroup morphism from $EK(A, B)$ to a semigroup $EK_\omega(A, B)$ which is constructed by *-monomorphisms from A into B_ω , where ω is a ultrafilter on \mathbf{N} and B_ω is the limit algebra $\ell_\infty(B)/C_\omega(B)$ which is also purely infinite simple. Then, applying the approximate intertwining argument in the sence of Elliott and the property of Kasparov product, Theorem B is obtained.

This note is a survey of Kirchberg's draft "The classification of purely infinite C*-algebras using Kasparov's Theory", and it is based on Dr. Rajarama Bhat's lecture and lecture note by Peter Ffris [13] in the program year (Sept., 1994 - Aug., 1995) in Operator Algebras and Applications at the Fields Institute. The author also has been writing the more detail explanation about it [22].

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2 Basic results on purely infinite simple C^* -algebras

Definition 2.1 *A projection p is called infinite if p is Murray-von Neumann equivalent to a proper subprojection of p*

For projection p, q , let us write $p \preceq q$ if p is equivalent to a subprojection of q .

Definition 2.2 *A simple C^* -algebra A is purely infinite(= pi) if every hereditary C^* -subalgebra contains an infinite projection.*

In the rest part of this note a "pi algebra" is simple in particular.

Example 2.3 *The Cuntz algebra O_n ($n = 2, 3, \dots, \infty$) is typical examples of simple and purely infinite [7].*

Proposition 2.4 (Cuntz[9], Kirchberg[17]) *Let A be a simple C^* -algebra ($A \neq \mathbf{C}, 0$). Then the following conditions are equivalent:*

- (i) *A is pi.*
- (ii) *For positive elements $a, b \in A$ with $\|a\| = \|b\| = 1$ and $\epsilon > 0$ there exists $c \in A$ with $\|c\| = 1$ such that $\|b - c^*ac\| < \epsilon$.*

When A has a unit, we get the following characterization.

Proposition 2.5 (Cuntz [9]) *Let A be a simple unital C^* -algebra. Suppose that A is not the scalars. Then the following conditions are equivalent:*

- (i) *A is a pi algebra.*
- (ii) *For any non-zero a in A there are elements x and y such that*

$$1 = xay.$$

- (iii) *For any non-zero positive element a in A and $\epsilon > 0$ there is an element x such that*

$$1 = xax^* \quad \text{and} \quad \|x\| \leq \|a\|^{-\frac{1}{2}} + \epsilon.$$

Corollary 2.6 (i) every hereditary C^* -subalgebra of a pi algebra is pi .

(ii) $A \otimes \mathbf{K}$ is pi if A is a pi algebra, where \mathbf{K} is a C^* -algebra generated by all compact operators on some Hilbert space.

(iii) A is a pi algebra if (and only if) A is simple and contains in multiplier algebra $M(A)$ a central sequence (for elements in A) of unital copies of E_2 (or of O_2), where E_2 is the C^* -subalgebra generated by s_1, s_2 of $O_3 = C^*(s_1, s_2, s_3)$.

(iv) (Rørdam [28], Lin [18]) Let A be a σ -unital C^* -algebra.

If $M(A)/A$ is simple, then $A \cong \mathbf{K}$ or A is pi .

Proof. (iii): Let a and b be positive elements in A with $\|a\| = \|b\| = 1$ and $\epsilon > 0$. Since A is simple there are x_1, \dots, x_n in A such that $\|b - \sum x_i^* a x_i\| < \epsilon$. From assumption there are isometries s_1, \dots, s_n such that $s_i^* s_j = \delta_{i,j}$ for all i, j and $\|s_i a - a s_i\| < \frac{1}{\sum_{i=1}^n \|x_i\|^2} \epsilon$ for i . Put $c = \sum s_i x_i$ and $p = \sum s_j s_j^*$, then

$$\begin{aligned} \|c^* p a p c - b\| &\leq \|c^* p a p c - c^* (\sum s_j a s_j^*) c\| \\ &\leq \|c\|^2 \|\sum s_j s_j^* a s_k s_k^* - \sum s_j s_j^* s_k a s_k^*\| \\ &\leq \|c\|^2 \|\sum_j s_j s_j^*| \sum_k (a s_k - s_k a) s_k\| \\ &\leq (\sum \|x_i\|^2) \frac{1}{\sum_{i=1}^n \|x_i\|^2} \epsilon = \epsilon. \end{aligned}$$

Hence, if we take ϵ as a small, A is purely infinite (see proof of (ii) \rightarrow (i) in Proposition 2.4).

For a unital C^* -algebra A we denote by $U(A)$ the group of unitaries of A , and by $U_0(A)$ the connected component of the unit.

Proposition 2.7 (Cuntz[9]) Let A be a unital pi algebra. Then:

(i) For every $g \in K_0(A)$ there is a non-zero projection p such that $[p] = g$.

(ii) $K_0(A)^+ = K_0(A)$.

(iii) If p and q are non-zero projections then $[p] = [q]$ iff $p \sim q$.

(iv) The canonical map $U(A)/U_0(A) \rightarrow K_1(A)$ is an isomorphism.

3 The generalized Weyl-von Neumann Theorem

In this section we present the Weyl-von Neumann type theorem along lines of Voiculescu and Kasparov (see [1],[14],[33]), which plays an important role in [17].

Definition 3.1 For a unital C^* -algebra A which has a unital copy of O_2 and a fixed pair of generators s_1, s_2 of O_2 , we define the Cuntz addition $\oplus = \oplus_{s_1, s_2}$ in A as follows:

$$a \oplus b = s_1 a s_1^* + s_2 b s_2^*, \quad a, b \in A.$$

Remark 3.2 Upto unitary equivalence, Cuntz addition is independent of which particular copy of O_2 we take: If s_1, s_2 and t_1, t_2 are generators of two copies of O_2 in A , then

$$a \oplus_{s_1, s_2} b = u(a \oplus_{t_1, t_2}) u^* \quad \text{for all } a, b \in A,$$

where $u = s_1 t_1^* + s_2 t_2^*$ is a unitary.

Theorem 3.3 (Kircberg [17]) Let A be a σ -unital C^* -algebra. Then the following properties of A are equivalent:

(i) $A \otimes \mathbf{K} \cong \mathbf{K}$ or there exists a unital π algebra B such that $A \otimes \mathbf{K} \cong B \otimes \mathbf{K}$.

(ii) For every unital separable C^* -algebra C of $M(A \otimes \mathbf{K})$ and every weakly nuclear unital completely positive map $V : C \rightarrow M(A \otimes \mathbf{K})$ with $V(C \cap (A \otimes \mathbf{K})) = \{0\}$ there exists a sequence of isometries $s_n \in M(A \otimes \mathbf{K})$ with

$$\begin{aligned} \alpha) s_n^* d s_n - V(d) &\in A \otimes \mathbf{K}, \forall d \in C \\ \beta) \lim_{n \rightarrow \infty} \|s_n^* d s_n - V(d)\| &= 0, \forall d \in C. \end{aligned}$$

(iii) For every unital separable C^* -algebra C of $M(A \otimes \mathbf{K})$ and every weakly nuclear unital representation $h : C \rightarrow M(A \otimes \mathbf{K})$ with $h(C \cap (A \otimes \mathbf{K})) = \{0\}$ there exists a sequence of unitaries $u_n \in M(A \otimes \mathbf{K})$ with

$$\begin{aligned} \alpha) u_n^* d u_n - d \oplus h(d) &\in A \otimes \mathbf{K}, \forall d \in C \\ \beta) \lim_{n \rightarrow \infty} \|u_n^* d u_n - (d \oplus h(d))\| &= 0, \forall d \in C. \end{aligned}$$

The following proposition is a key result to prove Theorem 3.3 (i) \rightarrow (ii).

Proposition 3.4 (Kirchberg [17]) Let A be a π algebra and let B be a separable C^* -algebra of A . Let Ω be a compact Hausdorff space. Then for every nuclear map $\phi : B \rightarrow C(\Omega, A)$ there exists a sequence (d_n) of contractions in $C(\Omega, A)$ and a sequence (h_n) in A^+ with $\|h_n\| = 1$ such that $\lim_{n \rightarrow \infty} \|h_n d_n\| = 0$ and

$$\phi(b) = \lim_{n \rightarrow \infty} d_n^* b d_n, \forall b \in B.$$

We shall prove the assertion in the case that Ω is a single point, that is, Corollary 3.5. The proof of general case is almost the same.

Corollary 3.5 *Let A be a σ -unital π i algebra and let B be a separable C^* -algebra of $M(A)$. For every nuclear map $\phi : B \rightarrow A$ there exists a sequence (d_n) of contractions in A such that*

$$\phi(b) = \lim_{n \rightarrow \infty} d_n^* b d_n, \forall b \in B.$$

Proof.

Let ϕ be a nuclear map from B to A . Fix elements b_1, b_2, \dots, b_m and $\varepsilon > 0$. By the nuclearity of ϕ there are completely positive contractive maps $\phi' : B \rightarrow M_n$ and $\phi'' : M_n \rightarrow A$ such that

$$\|\phi(b_k) - \phi'' \circ \phi'(b_k)\| < \varepsilon, \quad 1 \leq k \leq m.$$

Since $B \subset A \subset A \otimes K(= C)$, ϕ' extends to C by Arveson's extension theorem (let us denote the extension by ϕ' again) and we can assume that $(\phi')^{**}(1_{C^{**}}) = 1$. Since A is infinite dimensional, there is a positive element $h = h_1 \otimes e_{1,1}$, $h_1 \in A^+$, $\|h_1\| = 1$ such that $\|\phi'(h)\| < \varepsilon$. Let ρ be a pure state on C .

Claim 1: There are contractions f_1, \dots, f_n in C such that $f_i^* f_j = 0$ for $i \neq j$, $\rho(f_i^* f_i) = 1$ for all i , and $\|\phi'(b_k) - F^{(k)}\| < \varepsilon$, $k = 1, \dots, m+1$, where $F_{i,j}^{(k)} = \rho(f_i^* b_k f_j)$ and $b_{m+1} = h$.

Proof of Claim 1.

Let $\pi : C \rightarrow B(H_\rho)$ be the GNS-representation induced by ρ with a cyclic vector η . Since C is simple and π is faithful, we have a map $\hat{\phi}' : \pi(C) \rightarrow M_n$ defined by $\hat{\phi}'(\pi(c)) = \phi'(c)$. Note that $\hat{\phi}'$ extends to $\pi(\tilde{C})$.

Using Glimm's lemma (Lemma 3.6), we can find orthogonal vectors x_1, \dots, x_n in H_ρ such that

$$\|\phi'(b_k) - F^{(k)}\| < \varepsilon$$

for $k = 1, \dots, m+1$, where $F_{i,j}^{(k)} = \langle \pi(b_k)x_j, x_i \rangle$.

By Kadison's transitivity theorem, there are elements f_1, \dots, f_n in C such that $\pi(f_i)\eta = x_i$ and it follows that

$$\begin{aligned} \langle \pi(b_k)x_j, x_i \rangle &= \langle \pi(b_k)\pi(f_j)\eta, \pi(f_i)\eta \rangle \\ &= \langle \pi(f_i^* b_k f_j)\eta, \eta \rangle \\ &= \rho(f_i^* b_k f_j) \end{aligned}$$

for all i, j .

End of the proof of Claim 1.

Since ϕ'' is a contraction, we get

$$\|\phi'' \circ \phi'(b_k) - \phi''(F^{(k)})\| < \varepsilon, \quad k = 1, \dots, m+1.$$

Note that the multiplier algebra $M(C)$ of C contains a unital copy of O_n .

Claim 2: There exist e_1, \dots, e_n in C such that $\phi''(F^{(k)}) = \sum_{i,j} F_{i,j}^{(k)} e_i^* e_j$ for $k = 1, \dots, m+1$.

Proof of Claim 2.

Let $e_{i,j}$ be a canonical matrix units for M_n . Then, $[e_{i,j}]$ is a positive matrix in M_n . Since ϕ'' is completely positive, $[\phi''(e_{i,j})] (= G)$ is positive. Now define e_1, \dots, e_n in C by

$$[e_1, \dots, e_n] = [s_1, \dots, s_n] G^{\frac{1}{2}},$$

where s_1, \dots, s_n are generators of O_n in $M(C)$. Then, $G = [e_1, \dots, e_n]^* [e_1, \dots, e_n]$ and hence for each $[\alpha_{i,j}] \in M_n$ we have

$$\phi''([\alpha_{i,j}]) = \sum \alpha_{i,j} \phi''(e_{i,j}) = \sum_{i,j} \alpha_{i,j} e_i^* e_j.$$

End of the proof of Claim 2.

Therefore,

$$\|\phi'' \circ \phi'(b_k) - \sum_{i,j} F_{i,j}^{(k)} e_i^* e_j\| < \varepsilon, \quad k = 1, \dots, m+1.$$

Choosing an approximate unit x for C such that $\|xe_i - e_i\|$ is sufficiently small for all $i = 1, \dots, n$, we can get

$$\|\sum_{i,j} F_{i,j}^{(k)} e_i^* e_j - \sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(xe_j)\| < \varepsilon, \quad k = 1, \dots, m+1.$$

Claim 3: There is a contraction $y \in C$ such that

$$\|\sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(xe_j) - \sum_{i,j} e_i^* y^* (f_i^* b_k f_j) y e_i\| < \varepsilon$$

for all $k = 1, \dots, m+1$.

Proof of Claim 3. From Lemma 3.7 and claim 1, there exists $y \in C$ such that $\|F_{i,j}^{(k)} x^2 - y^* (f_i^* b_k f_j) y\|$ are small enough for

$$\|\sum_{i,j} F_{i,j}^{(k)} (e_i^* x)(xe_j) - \sum_{i,j} e_i^* y^* (f_i^* b_k f_j) y e_i\| < \varepsilon$$

for $k = 1, \dots, m+1$.

End of the proof of Claim 3.

So, with $d = \sum_i f_i y e_i$ we have

$$\left\| \sum_{i,j} F_{i,j}^{(k)}(e_i^* x)(x e_j) - d^* b_k d \right\| < \varepsilon, \quad k = 1, \dots, m+1.$$

From the above arguments, we obtain that

$$\|\phi(b_k) - d^* b_k d\| < \varepsilon \quad k = 1, \dots, m.$$

Moreover, since $\|\phi'' \circ \phi'(h)\| < \varepsilon$, we get $\|d^* h d\| < 4\varepsilon$. Hence, $\|h d\|$ can be made as small as we like.

If we replace d by $(1 \otimes e_{1,1})d(1 \otimes e_{1,1})$, then we get the assertion.

We put some observations about completely positive maps which were used in the proof of Corollary 3.5.

Lemma 3.6 (Glimm [1]) *Let $A \subset B(H)$ be a unital C^* -algebra. Suppose that $\phi : A \rightarrow M_n$ is a completely positive map and annihilates $A \cap K$. Then there is a net (v_λ) of operators : $C^n \rightarrow H$ such that*

$$\|\phi(a) - v_\lambda^* a v_\lambda\| \rightarrow 0$$

for all $a \in A$. If ϕ is unital, then the v_λ 's can be chosen to be isometries.

Lemma 3.7 (Kirchberg[17]) *Let A be a pi algebra, let ρ be a pure state on A and let $x \in A$ be a contraction. Then for $\delta > 0$ and compact subset K of A there exists a contractions $y \in A$ such that*

$$\|\rho(a)x^*x - y^*ay\| < \delta$$

for all $a \in K$.

4 Proof of Theorem A (1)

In this section we present the sketch of the proof of Theorem A (1), which is used to construct a semigroup $EK_\omega(B, A)$.

The following two results are key points in Theorem A.

Lemma 4.1 (Glimm [24]) *Let A be a non-type I, separable, unital C^* -algebra. Then, there are a C^* -subalgebra B of A and a closed left ideal L of A such that*

$$\begin{cases} B + L \cap L^* = N(L) \\ L + L^* + N(L) = A \\ N(L)/L \cap L^* \cong M_{2^\infty}, \end{cases}$$

where $N(L) = \{a \in L : La + La^* \subset L\}$.

Proof. From [20, Theorem 6.7.3] there is a C^* -subalgebra B of A and a closed projection q in A^{**} , commuting with B , such that $qAq = qB$ and qB is isomorphic to M_2^∞ .

Put $L = A^{**}(1 - q) \cap A$, then L is a closed left ideal as required.

Theorem 4.2 (Kirchberg [16]) *Let A be a separable unital C^* -algebra. Then A is exact if and only if there is a unital C^* -subalgebra C of M_{2^∞} and a closed two-sided AF-ideal J of C such that A is $*$ -isomorphic to C/J .*

We say that a subalgebra B of a C^* -algebra A is essential if it has no left annihilators in A , i.e. if $aB = \{0\}$ implies $a = 0$ for a in A .

An ideal I of A is essential if and only if it has non-trivial intersection with every non-trivial ideal of A .

Remark 4.3 *When A is a Cuntz algebra O_2 , then $L \cap L^*$ in Lemma 4.1 is an essential hereditary algebra. In fact, if it is not essential, there exists a non-zero hereditary subalgebra K of O_2 such that $LK = \{0\}$. Let r be a open projection corresponding to K , then $r = rq = qr$, where q is a closed projection corresponding to L . Hence, K can be embedded into M_2^∞ . But, since K is purely infinite simple C^* -algebra, this is a contradiction.*

Proposition 4.4 *Let A be a separable, unital exact C^* -algebra. Then there exist a C^* -subalgebra E of O_2 and a closed two-sided ideal D of E such that*

- (i) D is an essential hereditary subalgebra of O_2 .
- (ii) $E/D \cong A$.

Proof. From Glimm's theorem there is a closed left ideal L_1 of O_2 such that $L_1 + L_1^* + N(L_1) = O_2$ and $N(L_1)/L_1 \cap L_1^* \cong M_{2^\infty}$. Let q be a closed projection corresponding to L_1 . Note that q is a identity of M_{2^∞} and commute with elements in $N(L_1)$ (cf. $N(L_1) = \{(1 - q)O_2^{**}(1 - q) + qO_2^{**}q\} \cap O_2$). From Theorem 4.2, there is a unital C^* -subalgebra C of M_{2^∞} and a closed two-sided AF-ideal J of C such that A is $*$ -isomorphic to C/J . Then, $D = \{d \in N(L_1) : qd, qd^* \in J\}$ and $E = \{d \in N(L_1) : qd, qd^* \in C\}$ satisfy conditions (i) and (ii) in the statement from the previous remark.

Lemma 4.5 *For any essential hereditary proper subalgebra D of O_2 we have*

$$D \cong O_2 \otimes K.$$

Proof. Since D is an hereditary subalgebra of O_2 , it is a pi algebra by Corollary 2.6(i), thus it is either unital or stable by [38]. If D has a unit p , then $(1-p)D = \{0\}$ and $1-p = 0$ because D is essential. So, $D \cong O_2$ and this is a contradiction. Therefore, D must be stable. By Brown's theorem [3] D is stable isomorphic to O_2 , hence $D \cong O_2 \otimes \mathbf{K}$.

Recall a few basic definitions from extension theory.

For every short exact sequence $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$, we consider the Busby diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & B & \rightarrow & E & \rightarrow & A \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & B & \rightarrow & M(B) & \xrightarrow{\pi} & Q(B) \rightarrow 0, \end{array}$$

where $Q(B)$ denotes the corona algebra $M(B)/B$. For $\phi, \psi \in Hom(A, Q(B))$, we write $\phi \approx \psi$ if there exists a unitary $u \in M(B)$ such that $\psi(a) = \pi(u)\phi(a)\pi(u^*)$, $a \in A$. An element $\phi \in Hom(A, Q(B))$ is called trivial if there exists an element $\hat{\phi}$ such that $\pi \circ \hat{\phi} = \phi$, i.e., if ϕ is liftable. Furthermore we write $\phi \sim \psi$ if there exist trivial elements $\tau_1, \tau_2 \in Hom(A, Q(B))$ such that $\phi \oplus \tau_1 \approx \psi \oplus \tau_2$.

We recall that $Ext(A, B)$ is the semigroup $Hom(A, Q(B))/\sim$, where the zero element is the class of trivial elements. By $Ext^{-1}(A, B)$ is denoted the group of invertible elements in $Ext(A, B)$, i.e. the classes of c.p. liftable elements of $Hom(A, Q(B))$ [1].

By what we have seen so far, there exists for every separable unital exact C^* -algebra A , an exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & D & \rightarrow & E & \xrightarrow{\alpha} & A \rightarrow 0 \\ & & \downarrow \cong & & \downarrow \beta & & \downarrow \tau \\ 0 & \rightarrow & O_2 \otimes \mathbf{K} & \rightarrow & M(O_2 \otimes \mathbf{K}) & \xrightarrow{\pi} & Q(O_2 \otimes \mathbf{K}) \rightarrow 0 \end{array}$$

where E is a subalgebra of O_2 and $D \cong O_2 \otimes \mathbf{K}$ is essential in O_2 . Then, since D is nuclear and E is exact, we know that this extension is a semisplit extension from Effros-Haagerup Lifting Theorem [11, Theorem 3.4] (see also [35, Remark 9.5]), i.e., there is a unital completely positive map $\tau : A \rightarrow E$ which is right inverse for $E \rightarrow A$. So, If τ is a Busby invariant of this extension, then $[\tau]$ is invertible in $Ext(A, O_2 \otimes \mathbf{K})$. Then, from the following fact we know that $[\tau] = 0$, hence there exist liftable elements τ_1 and τ_2 in $Hom(A, Q(O_2 \otimes \mathbf{K}))$ such that $\tau \oplus \tau_1 \cong \tau_2$.

Theorem 4.6 *$Ext(A, O_2 \otimes K) \cong KK^1(A, O_2 \otimes \mathbf{K})$ is trivial for every C^* -algebra A .*

Proof. Since id_{O_2} and $id_{O_2} \oplus id_{O_2}$ are homotopic on O_2 [9], $KK(O_2, O_2) = \{0\}$. Using the property of Kasparov product we get $KK^1(A, O_2 \otimes \mathbf{K}) = \{0\}$.

Proof of Theorem A (1)

Let A be a separable exact C^* -algebra. We may assume that A is unital. From the above argument, we have only to show that τ is liftable. Denote by C the image of E in $M(O_2 \otimes \mathbf{K})$ by β and define $\phi : C \rightarrow M(O_2 \otimes \mathbf{K})$ by

$$\phi = \hat{\tau}_1 \circ \tau^{-1} \circ \pi.$$

Then, ϕ is a unital weakly nuclear map such that $\phi(C \cap (O_2 \otimes \mathbf{K})) = \{0\}$.

Now we employ the generalized Weyl-von Neumann theorem (Theorem 3.3). This gives a unitary $u \in M(O_2 \otimes \mathbf{K})$ such that

$$\pi(\phi(c)) \oplus \pi(c) = \pi(u^*)\pi(c)\pi(u), c \in C.$$

Let $a \in A$. For every $e \in E$ such that $\alpha(e) = a$ we have $\pi(c) = \tau(a)$, where $c = \beta(e)$. Thus $\phi(c) = \hat{\tau}_1(a)$, hence for all a

$$\pi(u^*)\tau(a)\pi(u) = \tau_1(a) \oplus \tau(a) = \tau_2(a).$$

Therefore, $[\tau] = [\tau_2]$ in $Ext(A, O_2 \otimes \mathbf{K})$, and τ is liftable.

5 Limit algebras

Definition 5.1 A filter on \mathbf{N} is a set ω of subsets of \mathbf{N} satisfying the following conditions:

- (i) $\emptyset \notin \omega$.
 - (ii) $L_1 \cap L_2 \in \omega$ for all $L_1, L_2 \in \omega$.
 - (iii) $L \in \omega$ whenever $L' \subset L$ for some $L' \in \omega$.
- We say that ω is an ultrafilter if in addition it satisfies
- (iv) For every $L \subset \mathbf{N}$, either $L \in \omega$ or $L^c \in \omega$.
- or equivalently
- (iv)' ω is not properly contained in any other filter.

Note that for any ultrafilter ω , the intersection $\bigcap_{L \in \omega} L$ is either empty or contains exactly one element.

In order to avoid pathological behaviour, we usually restrict our attention to ultrafilter ω for which $\bigcap_{L \in \omega} L = \emptyset$. Such ultrafilters are called *free*. Note that $\omega \in \beta\mathbf{N}$ is free if and only if $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$.

Definition 5.2 Let ω be an (ultra)filter and let A be a C^* -algebra. A sequence (a_n) in A said to converge to an element $a \in A$ along ω (written $a_n \xrightarrow{\omega} 0$ or $\lim_{\omega} a_n = 0$) if for $\varepsilon > 0$ there is an $L \in \omega$ such that $\|a - a_n\| < \varepsilon$ for all $n \in L$.

Let A be a unital C^* -algebra and ω be a ultrafilter on \mathbb{N} . Let $\ell_{\infty}(A) = \{(a_n) : a_n \in A, \sup \|a_n\| < +\infty\}$ and $c_{\omega}(A) = \{(a_n) \in \ell_{\infty}(A) : a_n \xrightarrow{\omega} 0\}$. Then $c_{\omega}(A)$ is a closed two-sided ideal in $\ell_{\infty}(A)$. Set $A_{\omega} = \ell_{\infty}(A)/c_{\omega}(A)$, and let π_{ω} be the quotient mapping $\ell_{\infty}(A) \rightarrow A_{\omega}$. We call A_{ω} the limit algebra of A . Note that the quotient map π_{ω} satisfies

$$\|\pi_{\omega}(a)\| = \lim_{\omega} \|a_n\|$$

, where the crucial point is that the limit always exists.

Remark 5.3 If ω is free, then

$$c_0(A) = \{(a_n) | \lim a_n = 0\} \subset c_{\omega}(A).$$

Hence A_{ω} is a quotient of $A_{\infty}(= \ell_{\infty}(A)/c_0(A))$.

Proposition 5.4 Let A be a unital C^* -algebra. Then, A_{ω} is a pi algebra if and only if A is a pi algebra.

Proof. Suppose that A_{ω} is a pi algebra. Let x be a non-zero element in A , and consider a canonical image of x in A_{ω} . Since A_{ω} is pi , there are sequences $(y_n), (z_n)$ of A such that

$$(y_1 x z_1, y_2 x z_2, \dots) - (1, 1, \dots) \in C_{\omega}.$$

So, there is a non-zero set $L \in \omega$ such that $\|y_n x z_n - 1\| < 1$ for $n \in L$. Hence, there are $y, z \in A$ such that $yxz = 1$. This implies that A is a pi algebra from Proposition 2.5(ii).

Suppose that A is a pi algebra. Take a non-zero positive element $a \in A_{\omega}$ with $\|a\| = 1$, write $a = [(a_n)]$. Since $\|a\| = 1$ and ω is an ultrafilter, there is a non-zero set $L \in \omega$ such that $\|a_n\| > \frac{1}{2}$ for $n \in L$. Define \tilde{a}_n by

$$\tilde{a}_n = \begin{cases} \frac{a_n}{\|a_n\|} & n \in L \\ y & (\|y\| = 1) \quad n \notin L \end{cases}$$

Note that for any $\varepsilon > 0$ there is a non-zero subset L_{ε} of L such that $|\|a_n\| - 1| < \varepsilon$. Set $\tilde{a} = [(\tilde{a}_n)]$. Then, $\tilde{a} = a$ in A_{ω} .

Since $\|\tilde{a}_n\| = 1$, there is a $x_n \in A$ such that $\|x_n\| \leq 1 + \frac{1}{n}$ and $x_n^* \tilde{a}_n x_n = 1$ for any $n \in \mathbb{N}$. Set $x = [(x_n)] \in A_{\omega}$. Then, we know

that $x^*ax = x^*\tilde{a}x = 1$, and this implies that A_ω is a pi algebra from Proposition 2.5(iii).

The following is a limit version of Corollary 3.5.

Proposition 5.5 *Let ω be a free ultrafilter in \mathbf{N} . If A is a unital pi algebra, B a separable C^* -subalgebra of A_ω containing 1_A , and $V : B \rightarrow A_\omega$ is a nuclear unital completely positive map. Then there exists a nonunitary isometry $s \in A_\omega$ with $V(b) = s^*bs$ for $b \in B$.*

Proof. Take a compact subset Δ_B of B such that B is the closed linear span of Δ_B . Fix $0 < \varepsilon < 1$ and let Y be the set of $d = (d_k) \in A_\omega$ such that $\|h_k d_k\| \leq \varepsilon$ for some $(h_k) \in A_\omega$, $h_k \in A^+$, $\|h_k\| = 1$. By Proposition 3.5 we have

$$\inf_{d \in Y} \sup_{b \in \Delta_B} \|\phi(b) - d^*bd\| = 0.$$

Claim: There is an element $s \in Y$ such that $\phi(b) = s^*bs$ for all $b \in B$.

Since ϕ is unital, s is an isometry. Moreover, since $\|hs\| \leq \varepsilon < 1$ for some $h \in A_\omega^+$, $\|h\| = 1$, s is not a unitary.

Proof of Claim. We may prove the following result:

Let A be a C^* -algebra and let ω be an ultrafilter in \mathbf{N} . Let Z be a set of contractions $A \rightarrow A$ and let Z_ω denote the set of all sequences of elements of Z . Let Ω be a compact metric space and let f_1, f_2 be continuous maps $\Omega \rightarrow A_\omega$. Then the infimum

$$\mu = \inf_{g \in Z_\omega} \sup_{x \in \Omega} \|f_1(x) - g(f_2(x))\|$$

is attained by some $g \in Z_\omega$.

Proof.

Choose a sequence $(g^{(n)})$ in Z_ω such that

$$\sup_{x \in \Omega} \|f_1(x) - g^{(n)}(f_2(x))\| < \mu + \frac{1}{n}$$

for all n . Fix the lifting of $A_\omega \rightarrow \ell_\infty(A)$, $y \mapsto (y_k)$. By the compactness of Ω , there are sets $L_n \in \omega$ such that

$$\sup_{x \in \Omega} \|f_1(x)_k - g^{(n)}(f_2(x))_k\| < \mu + \frac{1}{n}, \quad k \in L_n.$$

We may assume that $L_1 \supset L_2 \supset \dots$. Now let g be the diagonal element given by

$$g_k = \begin{cases} g_k^{(1)}, & \text{if } k \in \mathbf{N} \setminus L_1 \\ g_k^{(n)} & \text{if } k \in L_n \setminus L_{n+1} \end{cases}$$

Then for any $n \geq 2$ $\sup_{x \in \Omega} \|f_1(x)_k - g(f_2(x))_k\| < \mu + \frac{1}{n}$ for all $k \in L_n$. But then $\sup_{x \in \Omega} \|f_1(x) - g(f_2(x))\| = \mu$ as desired.

We can construct more general limit algebra as follows:

Let X be a locally compact Hausdorff space, let $C_b(X, A)$ be a C^* -algebra of C^* -algebra A -valued bounded continuous functions, and let $C_0(X, A)$ be a C^* -algebra of C^* -algebra A -valued continuous functions with vanishing at infinity. For every point ω in the corona space $\beta X \setminus X$ of X we consider a two-sided ideal in $C_b(X, A)$ consisting of all functions $f \in C_b(X, A)$ such that $\omega(\{x \in X \rightarrow \|f(x)\|\}) = 0$, where βX is a Stone-Ćech compactification of X .

Lemma 5.6 (R rdam[28]) *Let A be a simple unital C^* -algebra. Then, A is pi if and only if for any positive elements a, b in A with norm one there are c_1, c_2 in A such that $c_1ac_1^* + c_2ac_2^* = b$.*

Proof. Suppose that for any positive elements a, b in A with norm one there are c_1, c_2 in A such that $c_1ac_1^* + c_2ac_2^* = b$.

Define a continuous function $f_\varepsilon : \mathbb{R}^+ \rightarrow [0, 1]$ by

$$f_\varepsilon(t) = \begin{cases} 0 & t \in [0, \varepsilon] \\ 1 & t \in [2\varepsilon, \infty) \\ \text{linear} & t \in [\varepsilon, 2\varepsilon] \end{cases}$$

Let a be a positive element in A with norm one. Since $\overline{f_{\frac{1}{3}}(a)Af_{\frac{1}{3}}(a)}$ is simple and infinite dimensional, there are positive elements x', y' in $\overline{f_{\frac{1}{3}}(a)Af_{\frac{1}{3}}(a)}$ with norm one such that $x'y' = 0$.

Set $z = 1 - f_{\frac{1}{6}}(y')$, $y'' = f_{\frac{1}{3}}(y')$. Since A is simple, there is a unitary u in A such that

$$\overline{x'Ax'} \cap \overline{uy''Ay''}u^* \neq \emptyset.$$

Choose x in $\overline{x'Ax'} \cap \overline{uy''Ay''}u^* \neq \emptyset$. Put $y = u^*xu$. Then, x, y in \overline{aAa} , $xz = x$, and $yz = 0$.

By assumption there are c_1, c_2 in A such that $c_1xc_1^* + c_2xc_2^* = 1$. Put $t = c_1z + c_2u(1 - z)$. Then, $t(x + y)t^* = 1$. Since $x + y$ in \overline{aAa} , there is an infinite projection in \overline{aAa} which is equivalent to 1. Hence, A is purely infinite.

Proposition 5.7 *Let A be a unital C^* -algebra and ω in $\beta X \setminus X$. Then, $Q_\omega(A) = C_b(X, A)/J_\omega$ is a pi algebra if A is a pi algebra.*

Proof. Take positive elements $a, b \in Q_\omega(A)$ with $\|a\| = \|b\| = 1$. Then, we may assume that

$\|\tilde{a}(x)\| = \|\tilde{b}(x)\| = 1$ for all $x \in X$, where \tilde{a} and \tilde{b} are preimage in $C_b(X, A)_+$ of a and b , respectively.

Claim: For any compact set $\Omega \subseteq X$ and $\varepsilon > 0$ there is a continuous function l in $C(\Omega, A)$ with $\|l\| \leq 1$ such that $\|\tilde{b}|_\Omega - l^* \tilde{a}|_\Omega\| < \varepsilon$.

Proof. Take a state γ of $C^*\{1, \tilde{a}|_\Omega\}$ ($= C$) such that $\gamma(\tilde{a}|_\Omega) = 1$. Define a map V from C into $C(\Omega, A)$ by $V(f) = \gamma(f)\tilde{b}|_\Omega$ for $f \in C$. Then, V is completely positive contractive. Hence, there exists a contraction l in $C(\omega, A)$ such that $\|\tilde{b}|_\Omega - l^* \tilde{a}|_\Omega\| < \varepsilon$ from Proposition 3.4.

(End of the proof of claim)

Since ω is free ultrafilter, there is a sequence of open sets X_i of X such that $X_i^c \in \omega$, $\Omega_i = \overline{X_i} \subseteq X_{i+1}$, and $X = \bigcup_i X_i$. Then, take continuous functions $g_i : X \rightarrow [0, 1]$ such that $g_i|_{\Omega_i} = 1$ and $g_i|_{\Omega_{i+1}^c} = 0$. From the claim there are contractions $l_i \in C(\Omega_i, A)$ such that $\|l_i(x)^* \tilde{a}(x) l_i - \tilde{b}(x)\| < \frac{1}{2^i}$ for any $x \in \Omega_i$.

Set

$$\begin{aligned} c_1 &= g_1^{\frac{1}{2}} l_1 + (g_3 - g_2)^{\frac{1}{2}} l_3 + (g_4 - g_3)^{\frac{1}{2}} l_5 + \dots \\ c_2 &= (g_2 - g_1)^{\frac{1}{2}} l_2 + (g_3 - g_2)^{\frac{1}{2}} l_4 + \dots \end{aligned}$$

Then,

$\|\pi_\omega(c_1^* \tilde{a} c_1 + c_2^* \tilde{a} c_2 - \tilde{b})\| = 0$, where π_ω is a canonical quotient map from $C_b(X, A)$ to $Q_\omega(A)$. Hence, $Q_\omega(A)$ is a *pi* algebra from the previous lemma.

6 Elliott's intertwining principle

In this section we shall give a brief introduction to this principle [12].

Fix two sequences

$$\begin{aligned} (1) & A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} A_4 \longrightarrow \dots \\ (2) & B_1 \xrightarrow{\psi_{1,2}} B_2 \xrightarrow{\psi_{2,3}} B_3 \xrightarrow{\psi_{3,4}} B_4 \longrightarrow \dots \end{aligned}$$

of separable C^* -algebras and $*$ -homomorphisms. And fix dense sequences $F_i \subset A_i$ and $G_i \subset B_i$, respectively, for $i \in \mathbb{N}$. Let $A = \lim A_i$ and $B = \lim B_i$ denote the corresponding inductive limit C^* -algebras and $\phi_i : A_i \rightarrow A$ and $\psi_i : B_i \rightarrow B$ be the canonical homomorphisms.

Lemma 6.1 (Elliott[12]) *Let $\{\delta_n\}$ be a sequence in $[0, \infty)$ such that $\sum_{n=1}^\infty \delta_n < \infty$. Let $\alpha_i : A_i \rightarrow B_i$, $i \in \mathbb{N}$ be $*$ -homomorphisms such that*

$$\|\psi_{i,i+1} \circ \alpha_i(x) - \alpha_{i+1} \circ \phi_{i,i+1}(x)\| < \delta_i,$$

whenever $x \in S_i$, where S_i is the finite set of A_i consisting of the images in A_i of the first i terms of the sequences F_1, \dots, F_{i-1} .

Then the sequence $\{\psi_l \circ \alpha_l \circ \phi_{i,l}(x)\}$, $l \geq i$, converges in B for each $x \in A_i$ and all $i \in \mathbf{N}$. Furthermore, there is a $*$ -homomorphism $\alpha : A \rightarrow B$ such that

$$\alpha(\phi_i(x)) = \lim \psi_l \circ \alpha_l \circ \phi_{i,l}(x),$$

$x \in A_i$, $i = 1, 2, \dots$.

Proof. Let $F_i = \{a_{i,1}, a_{i,2}, \dots\}$. From the assumption, $\{\psi_k \circ \alpha_k \circ \phi_{i,k}(a_{i,l})\}_{k=l}^\infty$ ($k \geq l \geq i$) is a cauchy sequence for $a_{i,l} \in A_i$. So, we can define $\alpha'_i : F_i \rightarrow B$ by $\alpha'_i(x) = \lim_{k \rightarrow \infty} \psi_k \circ \alpha_k \circ \phi_{i,k}(x)$. Since $\alpha'_{i+1} \circ \phi_{i,i+1} = \alpha'_i$ and the density of F_i in A_i , we obtain a $*$ -homomorphism $\alpha : A \rightarrow B$, which is required one.

Definition 6.2 By an Elliott's approximate intertwining between the sequences (1) and (2), we mean $*$ -homomorphisms $\alpha_i : A_i \rightarrow B_i$ and $\beta_i : B_i \rightarrow A_{i+1}$ such that

$$\begin{aligned} \|\alpha_{i+1} \circ \beta_i(y) - \psi_{i,i+1}(y)\| &< 2^{-i} \\ \text{for } y &\in T_i \\ \|\beta_i \circ \alpha_i(x) - \phi_{i,i+1}(x)\| &< 2^{-i} \\ \text{for } x &\in S_i, i \in \mathbf{N}, \end{aligned}$$

where S_i is the finite subset of A_i consisting of the images in A_i of the first i terms of the sequences F_1, \dots, F_{i-1} , and G_1, \dots, G_{i-1} , along all possible paths in the diagram. Similarly, T_i is the finite subset of B_i consisting of the images in B_i of the first i terms of the sequences F_1, \dots, F_i and G_1, \dots, G_{i-1} , along all possible paths in the diagram.

Theorem 6.3 (Elliott[12]) An approximate intertwining between (1) and (2) induces a $*$ -isomorphism between $A = \lim A_i$ and $B = \lim B_i$.

Proof. From the definition of an approximate intertwining it follows that

$$\begin{aligned} \|\psi_{i,k} \circ \alpha_i(x) - \alpha_k \circ \phi_{i,k}(x)\| &\leq 2^{-i+2}, \\ x &\in S_i \\ \|\phi_{i+1,k} \circ \beta_i(y) - \beta_{k-1} \circ \psi_{i,k-1}(y)\| &\leq 2^{-i+1}, \\ y &\in T_i. \end{aligned}$$

Therefore, there are $*$ -homomorphisms $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$ from Lemma 6.1. By using the original estimates from the definitions it is easily seen that α and β are inverses of each other.

Definition 6.4 (Rørdam[29]) Two $*$ -homomorphisms $\phi, \psi : A \rightarrow B$ between C^* -algebras A and B are said to be approximately unitary equivalent if for every finite subset F of A and every $\varepsilon > 0$ there is a unitary $v \in B$ (or \tilde{B} if B has no unit) so that

$$\|v\phi(a)v^* - \psi(a)\| < \varepsilon$$

for all $a \in F$.

7 Kirchberg's semigroup(discrete)

In this section we introduce Kirchberg's semigroup $(EK_\omega(B, A))$ and present basic observations. This notion (idea) is an important tool to get Theorem B. We shall prove that for every simple separable unital nuclear C^* -algebra B $B \otimes D_2 \cong D_2$, where $D_2 = O_2 \otimes O_2 \otimes O_2 \cdots$.

We use the following definitions and observations to define and study a semigroup $EK_\omega(B, A)$.

Let D be a C^* -algebra such that its multiplier algebra $M(D)$ contains a unital copy of O_2 , let B be a C^* -algebra and $h_1, h_2 : B \rightarrow D$ $*$ -homomorphisms.

We say that h_1 n -dominates h_2 if there exist $d_1, d_2, \dots, d_n \in M(D)$ such that $d_1^*d_1 + \dots + d_n^*d_n = 1$ and $h_2(\cdot) = d_1^*h_1(\cdot)d_1 + \dots + d_n^*h_1(\cdot)d_n$.

h_1 dominates h_2 if there is an isometry $s \in M(D)$ with $h_2(\cdot) = s^*h_1(\cdot)s$. We will write $h_2 \prec h_1$.

Remark 7.1 In this case, ss^* is in the relative commutant $h_1(B)' \cap M(D)$ of $h_1(B)$ in $M(D)$.

Proof. Set $q = ss^*$. Then, since $s^*h_1(b^*b)s - s^*h_1(b^*)ss^*h_1(b)s = 0$, $qh_1(b^*)h_1(b)q = qh_1(b^*)qh_1(b)q$. So, $qh_1(b^*)(1-q)h_1(b)q = 0$. Hence, $(1-q)h_1(b)q = 0$, and $h_1(b)q = qh_1(b)q$ for any $b \in B$. Therefore, $h_1(b)q = qh_1(b)$ for any $b \in B$.

The following are simple observations:

Lemma 7.2 (i) If h_1 dominates h_2 and $h_2(B)' \cap M(D)$ contains a unital copy of O_2 , then Cuntz addition (see Definition 5.1) $h_1 \oplus h_2$ and h_1 are unitary equivalent in $M(D)$.

(ii) If h_1 n -dominates h_2 and $h_1(B)' \cap M(D)$ contains a unital copy of E_2 , then h_1 dominates h_2 .

(iii) If $h_0(B)' \cap M(D)$ contains a unital copy of O_2 , then a set of unitary equivalence classes of the $*$ -homomorphisms $h : B \rightarrow M(D)$

which are dominated by h_0 forms a semigroup $S(h_0, B, D)$ under Cuntz addition.

We use $[h]$ as an unitary equivalence class of $h : B \rightarrow M(D)$.

Proof. (i): From the assumption there is an isometry $d \in M(D)$ such that $h_2 = d^* h_1 d$. Take a generator $\{s_1, s_2\}$ of O_2 which is contained in $h_2(B)' \cap M(D)$. Then, $h_1 \oplus_{s_1, s_2} h_2 = s_1 h_1 s_1^* + s_2 h_2 s_2^*$. Set $u = (1 - dd^*)s_1^* + ds_1 d^* s_1^* + ds_2 s_2^*$, then u is a unitary in $M(D)$ and $u(h_1 \oplus_{s_1, s_2} h_2)u^* = h_1$.

(ii): From the assumption there are n elements $\{d_i\}_{i=1}^n$ in $M(D)$ such that $h_2 = \sum_{i=1}^n d_i^* h_1 d_i$ and $\sum_{i=1}^n d_i^* d_i = 1$. Since $h_1(B)' \cap M(D)$ contains a unital copy of E_2 , it contains O_∞ . Take n elements $\{s_i\}_{i=1}^n$ of generators of O_∞ .

Set $d = \sum_{i=1}^n s_i d_i$. Then, d is an isometry in $M(D)$, and

$$\begin{aligned} h_2 &= \sum_{i=1}^n d_i^* h_1 d_i \\ &= \sum_{i,j} d_i^* s_i^* h_1 s_j d_j \\ &= d^* h_1 d. \end{aligned}$$

(iii): Note that the definition of $h \oplus_{O_2} k$ is independent from the choice of a unital copy of O_2 in $M(D)$. Take $[h_1], [h_2] \in S(h_0, B, D)$, and write $h_i = d_i^* h_0 d_i$ ($i = 1, 2$), where d_1, d_2 are isometries in $M(D)$.

Then,

$$\begin{aligned} h_1 \oplus_{\sigma_1, \sigma_2} h_2 &= \sum_{i=1}^2 \sigma_i d_i^* h_0 d_i \sigma_i^* \\ &= \sum_{i,j,k} \sigma_i d_i^* \sigma_i^* \sigma_j h_0 \sigma_j^* \sigma_k d_k \sigma_k^* \\ &= (\sum_i \sigma_i d_i^* \sigma_i^*) (\sum_j \sigma_j h_0 \sigma_j^*) (\sum_k \sigma_k d_k \sigma_k^*) \\ &= d^* h_0 d, \end{aligned}$$

where σ_1, σ_2 are generators of O_2 in $h_0(B)' \cap M(D)$ and $d = \sigma_1 d_1 \sigma_1^* + \sigma_2 d_2 \sigma_2^*$.

Hence, the unitary equivalence class of $h_1 \oplus h_2$ is contained in $S(h_0, B, D)$.

Proposition 7.3 *Let $h_0 : B \rightarrow M(D)$ be a $*$ -homomorphism so that $h_0(B)' \cap M(D)$ contains a unital copy of O_2 . Then*

- (1) *the set $G(h_0, B, D) = \{[h \oplus h_0] : h \prec h_0\}$ forms a subgroup of $S(h_0, B, D)$. Moreover,*
- (2) *$G(h_0, B, D)$ is isomorphic to the Grothendieck group of $S(h_0, B, D)$ ($= \text{Groth}(S(h_0, B, D))$).*

Proof. (1): From Lemma 7.2(i), we know that

$$[h_0] + [h_0] = [h_0 \oplus h_0] = [h_0],$$

so $[h_0]$ is a zero element in $G(h_0, B, D)$. Take nonzero element $[h \oplus h_0]$. Since $h \prec h_0$, there is an isometry d in $M(D)$ such that $h = d^*h_0d$. Set $k = (1 - dd^*)h_0 + dh_0d^*$. Then,

$$k \oplus_{s_1, s_2} h = s_1(1 - dd^*)h_0s_1 + s_1dh_0d^*s_1^* + s_2d^*h_0ds_2^*,$$

where s_1, s_2 are generators of a unital copy of O_2 in $M(D)$.

Set $u = \sigma_1as_2^* + \sigma_1(1 - dd^*)s_1^* + \sigma_2d^*s_1^*$, where σ_1, σ_2 are generators of O_2 in $h_0(B)' \cap M(D)$. Then, u is a unitary in $M(D)$, and

$$\begin{aligned} u(k \oplus_{s_1, s_2} h)u^* &= \sigma_1dd^*h_0dd^*\sigma_1^* + \sigma_1(1 - dd^*)h_0\sigma_1^* + \sigma_2h_0d^*d\sigma_2^* \\ &= \sigma_1h_0\sigma_1^* + \sigma_2h_0\sigma_2^* \\ &= h_0. \end{aligned}$$

Hence,

$$\begin{aligned} [k \oplus h_0] + [h \oplus h_0] &= [k \oplus h \oplus h_0 \oplus h_0] \\ &= [h_0 \oplus h_0 \oplus h_0] = [h_0]. \end{aligned}$$

Therefore, $G(h_0, B, D)$ is a group.

(2) Exercise.

Now we will define a semigroup $EK_\omega(B, A)$.

Definition 7.4 Let A be a unital π i in Cuntz standard form ($A = A^{st}$), that is, A contains a unital copy of O_2 . and let B be a separable unital C^* -algebra. Let ω be a free ultrafilter of \mathbb{N} , that is, $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

Then $EK_\omega(B, A)$ is a set of unitary equivalence classes $[h]$ of nuclear unital $*$ -monomorphisms $h : B \rightarrow A_\omega$.

Under the Cuntz addition, that is, $[h] + [k] = [h \oplus_{O_2} k]$, $EK_\omega(B, A)$ becomes an abelian semigroup. Note that if h, k are nuclear, then it is easily seen that $h \oplus_{O_2} k$ is also nuclear.

We show $EK_\omega(B, A)$ is a group if B is exact.

Lemma 7.5 Let A be a unital π i algebra with $A = A^{st}$ and let B be a unital separable exact C^* -algebra. Then for any $[h_1], [h_2] \in EK_\omega(B, A)$,

$$h_2 \prec h_1.$$

Proof. Let $C = h_1(B)$. Define $h : C \rightarrow A_\omega$ by $h = h_2 \circ h_1^{-1}$. Then, from Proposition 5.5 there is a proper isometry $s \in A_\omega$ such that $h(c) = s^*cs$ for $c \in C$. Put $c = h_1(b)$ ($b \in B$). Then, $h_2(b) = s^*h_1(b)s$ for any $b \in B$. This implies the conclusion.

Proposition 7.6 *Under the same assumption as in the previous lemma*

- (i) $EK_\omega(B, A)$ is a group.
- (ii) $[h] = 0$ in $EK_\omega(B, A)$ if and only if $h(B)' \cap A_\omega$ contains a unital copy of O_2 .

Proof. (i): Let h_0 be an inclusion map

$$h_0 : B \hookrightarrow O_2 \subseteq O_2 \otimes O_2 \subseteq O_2 \subseteq A \subseteq A_\omega$$

which is guaranteed by Theorem A. From the previous lemma we know that for any element $[h]$ in $EK_\omega(B, A)$ $h \prec h_0$. Note that $h_0(B)' \cap A_\omega$ contains a unital copy of O_2 .

Hence, $EK_\omega(B, A)$ is a group from Lemma 7.2(i) and Proposition 7.3.

(2): This comes from Lemm 7.2(iii).

Remark 7.7 *Let D_2 be a decoy of O_2 , that is, $O_2 \otimes O_2 \otimes \dots$. Then,*

$$EK_\omega(B, D_2) = 0$$

for any separable unital exact C^ -algebra B .*

Proof. Note that D_2 is a π algebra from the next lemma and Corollary 2.6. The statement comes from Corollary 7.9 and the freeness of ω .

Lemma 7.8 *C^* -algebra D_2 contains a central sequence of unital copies of O_2 . Hence, D_2 is a π algebra.*

Proof. Let Δ_D be a compact subset of D such that the linear span of elements in Δ_D is dense in D_2 . We have only to show that for any ε there is a generator $\{s_1, s_2\}$ of O_2 such that $\|s_i x - x s_i\| < \varepsilon$ ($i = 1, 2$) for any $x \in \Delta_D$. For each $x \in D_2$ and for any $\varepsilon > 0$ there is an element $y \in D_2$ such that

$y \in \underbrace{O_2 \otimes \dots \otimes O_2}_{n(x)} \otimes 1 \otimes \dots$. Since Δ_D is compact, there are n elements $x_1, \dots, x_n \in \Delta_D$ such that $\Delta_D \subseteq \bigcup_{i=1}^n U(x_i, \varepsilon)$. Set $k = \max\{n(x_i)\}$. Then, if we take a generator $\{s_1, s_2\}$ of O_2 from $\underbrace{1 \otimes \dots \otimes 1}_k \otimes O_2 \otimes \dots$, $\|x s_i - s_i x\| < 4\varepsilon$ ($i = 1, 2$) for all $x \in \Delta_D$.

Corollary 7.9 *Let A be a unital C^* -algebra and let $h : D_2 \rightarrow A$ be a unital $*$ -monomorphism. Then, $h(D_2)' \cap A_\omega$ contains a unital copy of O_2 .*

The following is a reformation of Theorem 6.3.

We call a C^* -algebra A π -sun algebra if A is purely infinite simple, separable, unital, nuclear algebra.

Lemma 7.10 *If A and B are π -sun algebra with $A = A^{st}, B = B^{st}$, and let $h : A \rightarrow B, k : B \rightarrow A$ unital $*$ -homomorphisms such that $[kh] = [id_A]$ in $EK_\omega(A, A)$ and $[hk] = [id_B]$ in $EK_\omega(B, B)$ then there exists an isomorphism $\phi : A \rightarrow B$ which is approximately unitary equivalent to h .*

Proof. Since $[kh] = [id_A]$, there is a unitary $u \in A_\omega$ such that $kh(a) = uau^*$ for $a \in A$. Since u can be lifted to a sequence $\{u_n\}$ of unitaries in $\ell_\infty(A)$, we may know that kh and id_A are approximately unitary equivalent. Similarly, we know that hk and id_B are approximately unitary equivalent. Let X_A and X_B be dense sequences in A and B , respectively.

Then we can find sequences of unitaries $\{u_n\}$ in A and $\{v_n\}$ in B such that $A_i = A, \phi_{i,i+1} = u_i(\cdot)u_i^*, B_i = B, \psi_{i,i+1} = (\cdot)v_i^*, \alpha_i = h, \beta_i = k, F_i = X_A$, and $G_i = X_B$ for all $i \in \mathbb{N}$ induces approximate intertwining in Definition 6.2. So, there is an isomorphism $\phi : A \rightarrow B$ by Theorem 6.3.

It is easily seen that ϕ is approximately unitary equivalent to h from the construction.

Proposition 7.11 *If A is a π -sun algebra with $A = A^{st}$ and $[id_A] = 0$ in $EK_\omega(A, A)$, then $A \cong D_2$.*

Proof. Let $h : A \hookrightarrow D_2 \subseteq A$ and $k : D_2 \hookrightarrow A$ be $*$ -monomorphisms which is guaranteed by Theorem A (note that D_2 is nuclear). Then $kh : A \rightarrow D_2 \rightarrow A \subseteq A_\omega$ and $kh(A)' \cap A_\omega$ contains a unital copy of O_2 from the previous corollary. Hence, $[kh] = 0$ in $EK_\omega(A, A)$. From the assumption $[kh] = [id_A]$.

On the contrary, since $[hk] \in EK_\omega(D_2, D_2) (= 0)$, we know that $[hk] = 0 = [id_{D_2}]$.

Hence, from Lemma 7.10 $A \cong D_2$.

Corollary 7.12 (i) *If B is simple separable unital nuclear and contains a central sequence of unital copies of O_2 , then $B \cong D_2$.*

(ii) *For every simple separable unital nuclear B , we have $B \otimes D_2 \cong D_2$.*

Proof. (i): Let Δ_B be a compact set of B such that linear span of elements in Δ_B is dense B . From the assumption there is a central sequence $\{s_1^j, s_2^j\}_{j=1}^\infty$ of unital copies of O_2 such that $\|[s_k^j, x]\| < \frac{1}{j}$ for any $j \in \mathbb{N}$ and $x \in \Delta_B$.

Note that B is π (Corollary 2.6(iii)) Now set $T_1 = (s_1^1, s_1^2, \dots)$ and $T_2 = (s_2^1, s_2^2, \dots)$, then $C^*(T_1, T_2) \cong O_2 \subseteq B_\omega$. Moreover, $id_B(B)' \cap B_\omega$ contains $C^*(T_1, T_2)$, hence $[id_B] = 0$ in $EK_\omega(B, B)$ from Lemma 7.2(iii). So, from Proposition 7.11 $B \cong D_2$.

(ii): Since D_2 has a central sequence of unital copies of O_2 , $B \otimes D_2$ has also the same property. Hence, from (i) $B \otimes D_2 \cong D_2$.

8 Kirchberg's semigroup(continous)

In this section we introduce a continuous version of $EK_\omega(B, A)$.

We assume that A is a unital π i algebra with $A = A^{st}$ and B is a separable unital exact C^* -algebra.

Definition 8.1 Let $\omega \in \beta(\mathbf{R}_+) \setminus \mathbf{R}_+$ is in the corona of \mathbf{R}_+ , there is a canonical epimorphism $\pi_\omega : C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A) \rightarrow A_\omega^{\mathbf{R}_+}$, where $A_\omega^{\mathbf{R}_+}$ is π i (Proposition 5.7)

$$C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A)/J_\omega/C_0(\mathbf{R}_+, A) (\cong C_b(\mathbf{R}_+, A)/J_\omega).$$

We call a unital $*$ -monomorphism

$$h : B \rightarrow C_b(\mathbf{R}_+, A)/C_0(\mathbf{R}_+, A) = Q(\mathbf{R}_+, A)$$

completely faithful if $\pi_\omega \circ h$ is a $*$ -monomorphism for every $\omega \in \beta\mathbf{R}_+ \setminus \mathbf{R}_+$.

h is constant if $h(B) \subseteq A \subseteq Q(\mathbf{R}_+, A)$.

h is D_2 -factorizable if $h(B)$ is contained in a unital copy of D_2 in $Q(\mathbf{R}_+, A)$.

h is scaling invariant if for every topological isomorphism σ of \mathbf{R}_+ onto \mathbf{R}_+ we have that h and $\hat{\sigma} \circ h$ are approximately unitary equivalent, where $\hat{\sigma}$ is the $*$ automorphism of $Q(\mathbf{R}_+, A)$ by σ .

From the above definitions we introduce the following three abelian semigroups under the Cuntz addition:

$$CEK(B, A) = \{[h] : h \text{ is constant unital nuclear } * \text{-monomorphism}\}.$$

$$EK(B, A) = \{[h] : h \text{ is completely faithful unital nuclear } * \text{-monomorphisms}\}.$$

$$SEK(B, A) = \{[h] : h \text{ is unital nuclear } * \text{-monomorphism}\}.$$

Remark 8.2 One has $CEK(B, A) \subseteq EK(B, A)$ and

$$SEK(B, A) + CEK(B, A) \subseteq EK(B, A) + SEK(B, A) \subseteq EK(B, A).$$

Proposition 8.3 (i) Any two completely faithful unital nuclear $*$ -monomorphisms from B into $Q(\mathbf{R}_+, A)$ 2-dominates each other.

(ii) Any two D_2 -factorizable $*$ -monomorphisms are unitarily equivalent, and their class is contained in $CEK(B, A)$. In particular, if

$$h_0 : B \hookrightarrow O_2 \subseteq O_2 \otimes O_2 \subseteq O_2 \subseteq A \subseteq Q(\mathbf{R}_+, A),$$

then $[h_0] = 2[h_0]$ in $SEK(B, A)$.

(iii) $SEK(B, A) + [h_0]$ is a subgroup of $EK(B, A)$ which is isomorphic to $Groth(SEK(B, A)) = Groth(EK(B, A))$.

(iv) If $0 = X_1 < X_2 < \dots$ is a sequence in \mathbf{R}_+ with $\lim_{n \rightarrow \infty} X_n = \infty$, ω is a free ultrafilter on \mathbf{N} and $\bar{\omega} \in \beta\mathbf{R}_+ \setminus \mathbf{R}_+$ defined by (X_n) and ω , then there is a canonical isomorphism from $A_{\bar{\omega}}^{\mathbf{R}_+}$ onto A_{ω} and $\pi_{\bar{\omega}} : Q(\mathbf{R}_+, A) \rightarrow A_{\bar{\omega}}^{\mathbf{R}_+} \cong A_{\omega}$ defines a semigroup morphism from $SEK(B, A)$ into $EK_{\omega}(B, A)$.

Proof. Hint: For the proof of (i) we use the following claim.

Claim. Let A be a unital π -algebra, X a locally compact σ -compact Hausdorff space, $1_A \in B \subseteq Q(X, A) = C_b(X, A)/C_0(X, A)$ a separable unital completely faithfully embedded C^* -subalgebra of $Q(X, A)$ and $V : B \rightarrow Q(X, A)$ a nuclear unital completely positive map.

Then there exists $a_1, a_2 \in Q(X, A)$ with $V(b) = a_1^* b a_1 + a_2^* b a_2$ for every $b \in B$.

Here $B \subseteq Q(X, A)$ is called completely faithfully embedded if any free ultrafilter ω in $\beta X \setminus X$ $\pi_{\omega}|B$ is faithful, where π_{ω} is a canonical quotient map from $Q(X, A)$ to $Q(X, A)/(J_{\omega} + C_0(X, A)) (\cong Q_{\omega}(A)$ in Proposition 5.7).

Corollary 8.4 *Let ω be a free ultrafilter in \mathbf{N} . Then, the natural map from $Groth(SEK(B, A))$ into $EK_{\omega}(B, A)$ is injective.*

Proposition 8.5 *If A and B are π -algebras with $A = A^{st}$ and $B = B^{st}$ and $h : A \rightarrow B$, $k : B \rightarrow A$ unital $*$ -homomorphisms such that $[kh] + [h_0^A] = [id_A] + [h_0^A]$ in $Groth(SEK(A, A))$ and $[hk] + [h_0^B] = [id_B] + [h_0^B]$ in $Groth(SEK(B, B))$, then there exists an isomorphism $\phi : A \rightarrow B$ which is approximately unitary equivalent to h .*

Proof. This comes from Lemma 7.10 and Corollary 8.4.

9 Proof of Theorem B

In this section we present the outline of Theorem B. If a reader is interested in the detail proof of main theorem (Theorem 11.1), he or she may try to read (or complete) arguments in [17, Appendix].

The following is a main theorem in an article of Kirchberg.

Theorem 9.1 *Let B be a unital separable and exact C^* -algebra such that B contains a unital copy of O_2 and A a π -algebra with $A = A^{st}$, then there is a group isomorphism ϕ from $SEK(B, A) + [h_0] = Groth(SEK(B, A))$ onto $KK_{nuc}(B, A)$.*

Here $KK_{nuc}(B, A)$ is a group which was studied by Skandalis [31]:
(Note that if B is nuclear, then $KK_{nuc}(B, A)$ is a usual KK -group $KK(B, A)$)

Let's see the outline of Theorem 9.1.

Let $Q(X, D) = C_b(X, D)/C_0(X, D)$ for a C^* -algebra D and a locally compact Hausdorff space X . We define a new abelian semigroup $ES(B, A)$ as a set of unitary equivalence classes of nuclear*-monomorphisms $h : B \otimes K \rightarrow Q(\mathbf{R}_+, A \otimes K)$. Then, we define a map τ from $SEK(B, A)$ to $ES(B, A)$ by

$$\tau([h]) = [h \otimes id_K].$$

Proposition 9.2 *Let A be a π_1 algebra with $A = A^{st}$ and let B be a separable unital exact C^* -algebra that contains a unital copy of O_2 . Then, τ is a semigroup isomorphism.*

Next, we define one more semigroup $SExt_{nuc}(B, A)$ as a set of unitary equivalence classes of nuclear *-monomorphisms $h : B \otimes K \rightarrow M(C_0(\mathbf{R}, A \otimes K)/C_0(\mathbf{R}, A \otimes K))$. Define a map ψ from $ES(B, A)$ to $SExt_{nuc}(B, A)$ by

$\psi([h]) = [(h_0, h)]$, where h_0 is an absorbing element from $B \otimes K$ into $Q(\mathbf{R}_-, A \otimes K)$ and (h_0, h) is a nuclear *-monomorphism from $B \otimes K$ into $Q(\mathbf{R}_-, A \otimes K) \oplus Q(\mathbf{R}_+, A \otimes K) \cong Q(\mathbf{R}, A \otimes K) \subset M(C_0(\mathbf{R}, A \otimes K)/C_0(\mathbf{R}, A \otimes K))$.

Proposition 9.3 *Under the same assumption in the previous proposition ψ maps $ES(B, A)$ onto the absorbing classes of $SExt_{nuc}(B, A)$, that is, $SExt_{nuc}(B, A) + [h_0] (\cong Groth(SExt_{nuc}(B, A)))$. Moreover, this induces a group isomorphism from $Groth(ES(B, A))$ from $Groth(SExt_{nuc}(B, A))$.*

From the construction we know that if B is nuclear, then $Groth(SExt_{nuc}(B, A))$ is a usual extension group $Ext^{-1}(B \otimes K, C_0(\mathbf{R}, A \otimes K))$. On the contrary, from the KK -Theory there is a correspondence between $Ext^{-1}(B \otimes K, C_0(\mathbf{R}, A \otimes K))$ and $KK(B, A)$. Hence, if B is nuclear, then there is a group isomorphism from $SEK(B, A) + [h_0]$ to $KK(B, A)$. If B is a general exact C^* -algebra, see [31].

Remark 9.4 *Since $KK_{nuc}(B, A)$ is the homotopy invariant, we have*

$$\phi([\hat{\sigma} \circ h]) = \phi([h]) \text{ for any topological isomorphism } \sigma \text{ from } \mathbf{R}_+ \text{ to } \mathbf{R}_+.$$

Hence,

$$[\hat{\sigma} \circ (h \oplus h_0)] = [\hat{\sigma} \circ h] + [h_0] = [h] + [h_0] = [h \oplus h_0].$$

Corollary 9.5 (1) $O_2 \cong D_2$.

(2) $B \otimes O_2 \cong O_2$ for any simple separable unital nuclear C^* -algebra B .

Proof. (1): Since $KK(O_2, O_2) = 0$, $Groth(SEK(O_2, O_2)) = 0$ from Theorem 9.1. Hence, $O_2 \cong D_2$ from Corollary 8.4(ii) and Proposition 7.11.

(2): This comes from (1) and Corollary 7.12(ii).

Lemma 9.6 *A nuclear $*$ -monomorphism $h : B \rightarrow Q(\mathbf{R}_+, A)$ is scaling invariant (modulo unitary equivalence) if and only if h is unitary equivalent to a constant $*$ -monomorphism $k : B \rightarrow A \subseteq Q(\mathbf{R}_+, A)$.*

Corollary 9.7 *If B is separable unital nuclear C^* -algebra with a unital copy of O_2 , and A is a π -algebra with $A = A^{st}$, then every element $z \in KK_{nuc}(B, A)$ is of the form $\phi([h \oplus h_0])$, where h is a unital nuclear $*$ -monomorphism from B into A .*

Theorem B

Let A and B be π -sun algebras with $A = A^{st}$ and $B = B^{st}$. If $z \in KK(A, B)$ is a KK -equivalence, then there exists an isomorphism ψ from A onto B such that $\phi([\psi] + [h_0]) = z$ in $KK(A, B)$.

Proof. From assumption there is an inverse y of z in $KK(B, A)$ such that $zy = Id_B$ and $yz = Id_A$. From the previous result, there are nuclear $*$ -monomorphisms $h : A \rightarrow B$ and $k : B \rightarrow A$ such that $\phi([h \oplus h_0^A]) = z$ and $\phi([k \oplus h_0^B]) = y$.

Using Kasparov product we get

$$[hk] + [h_0^B] = [id_B] + [h_0^B]$$

$$[kh] + [h_0^A] = [id_A] + [h_0^A].$$

Hence, from Proposition 8.5 there exists an isomorphism ψ from A to B which is approximately unitary equivalent to h . Therefore, they induce the same class in $EK_\omega(B, A)$, where ω is a free ultrafilter on \mathbf{N} , that is, $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$. Therefore, we get $[\psi] = [h]$ in $SEK(B, A)$ from Corollary 8.4

Corollary 9.8 *Let A and B be π -sun algebras.*

(1) *A and B are KK -equivalent if and only if they are stable isomorphic.*

(2) *If there exists KK -equivalence x in $KK(A, B)$ with*

$$\gamma_0(x)([1_A]_0) = [1_B]_0,$$

then A and B are isomorphic, where γ_0 is a natural map from $KK(A, B)$ to $Hom(K_0(A), K_0(B))$ which is induced by Kasparov product.

Proof. (1): Take projections p in A and q in B such that $[p]_0 = 0$ in $K_0(A)$ and $[q]_0 = 0$ in $K_0(B)$. Then, both pAp and qBq are in Cuntz standard form.

Since pAp and qBq are KK-equivalent, there are isomorphic from Theorem B. So, A and B are stable isomorphic from [2].

(2): As in the same argument we have an isomorphism τ from pAp to qBq for some projections $p \in A$ and $q \in B$ such that $\tau_0([1_A]_0) = [1_B]_0$. Take projections $p_1 \in pAp$ and $q_1 \in qBq$ so that p_1 is equivalent to 1_A , and q_1 is equivalent to 1_B .

Since $\tau_0([p_1]_0) = [q_1]_0$, there is a partial isometry u in qBq such that $u^*u = \tau(p_1)$ and $uu^* = q_1$. So, p_1Ap_1 is isomorphic to q_1Bq_1 , hence A is isomorphic to B .

Corollary 9.9 *Let A be a simple separable unital nuclear C^* -algebra. Then, A is pi if and only if $A \cong A \otimes O_\infty$.*

Proof. Let $\phi : A \rightarrow A \otimes O_\infty$ be a $*$ -homomorphism defined by $\phi(a) = a \otimes 1$. Then, this induces KK-equivalence in $KK(A, A \otimes O_\infty)$. Note that $\phi_0([1_A]_0) = [1_{A \otimes O_\infty}]_0$. Therefore, A and $A \otimes O_\infty$ are isomorphic from Corollary 9.8(2).

Corollary 9.10 *If A and B are pi-sun algebras satisfying the UCT, and if $\sigma : K_*(A) \rightarrow K_*(B)$ is an isomorphism with $\sigma_0([1_A]_0) = [1_B]_0$, then there exists an isomorphism τ from A onto B with $K_*(\tau) = \sigma_*$.*

Proof. This comes from the same argument as in Corollary 9.9.

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